*Vy*ZX

Formal Verification of a Graphical Language

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The ZX Calculus...

... is a graphical language for reasoning about quantum systems

ZX diagrams are open graphs consisting of green "Z" or red "X" "spiders" and "connections" between them

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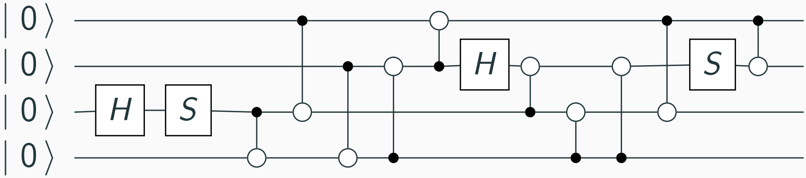
Used for compilation, simulation, error correction & more

Key benefit: Diagrammatic rewrites complete & more comprehensible than circuits or matrices

Example: Entanglement (Van de Wetering)



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$$[1 \ 0 \ 0 \ 0 \ i \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ i]^\top$$

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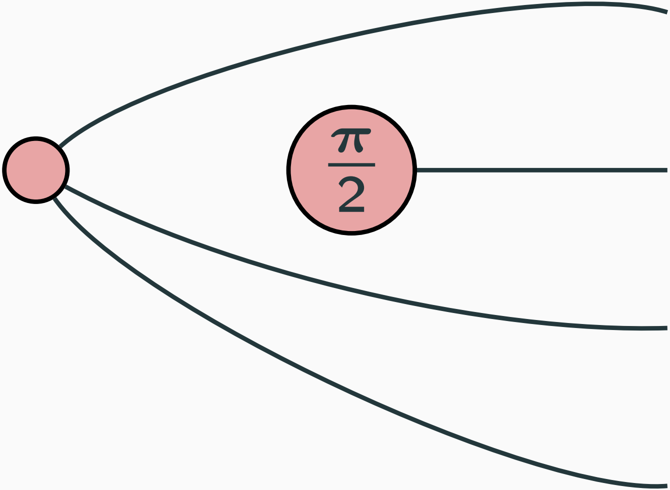

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- $|+\rangle, |-\rangle$ represents the X basis states (transform with hadamard (H))

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- The bra notation $\langle\psi|$ is equivalent to $|\psi\rangle^T$.
- When measuring a qubit, it collapses to either $|0\rangle$ or $|1\rangle$.
- The probability of measuring $|0\rangle$ is $|\alpha|^2$.
- The probability of measuring $|1\rangle$ is $|\beta|^2$.
- The sum of probabilities is always 1: $|\alpha|^2 + |\beta|^2 = 1$.

Quantum Circuit Model

- Quantum operations are represented as gates in a circuit model.
- Each gate acts on qubits, changing their states.
- Common gates include Hadamard (H), Pauli-X (X), Pauli-Y (Y), and Pauli-Z (Z).
- Quantum circuits are read from left to right, and gates are applied in sequence.

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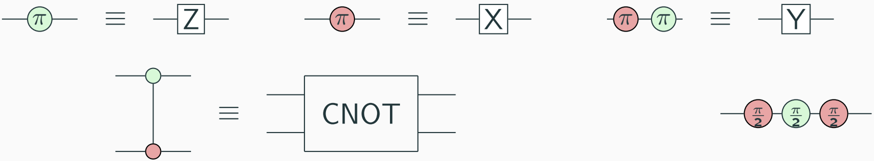
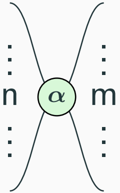
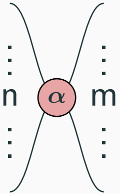


Diagram Semantics



$$= \quad \overset{n}{|0\rangle\cdots|0\rangle} \mapsto \overset{m}{|0\rangle\cdots|0\rangle}$$

$$\overset{n}{|1\rangle\cdots|1\rangle} \mapsto e^{i\alpha}\overset{m}{|1\rangle\cdots|1\rangle}$$



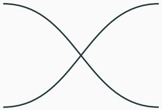
$$= \quad \overset{n}{|-\rangle\cdots|-\rangle} \mapsto \overset{m}{|-\rangle\cdots|-\rangle}$$

$$\overset{n}{|+\rangle\cdots|+\rangle} \mapsto e^{i\alpha}\overset{m}{|+\rangle\cdots|+\rangle}$$

Diagram Semantics

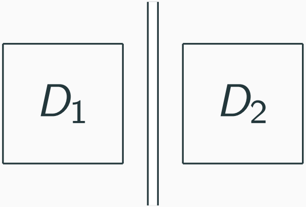
$$\mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$$

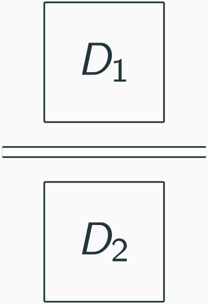


$$\mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Diagram Semantics

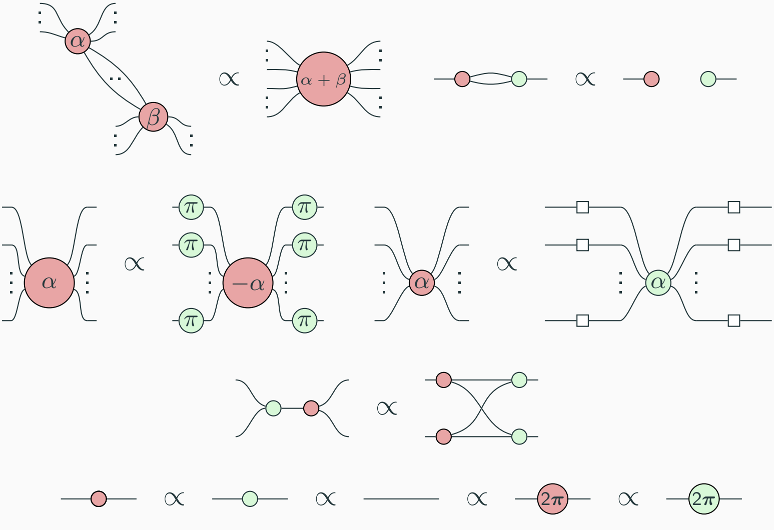


$$\mapsto \quad [D_2] \times [D_1]$$



$$\mapsto \quad [D_1] \otimes [D_2]$$

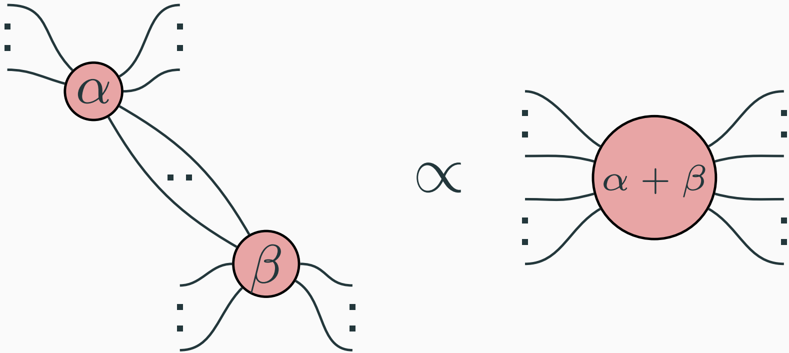
Rewriting Diagrams



Spider Fusion, The Hopf Rule, the Bi-pi Rule (Pi-Copy), Bi-hadamard Rule, the Bialgebra rule, and the identity rule

Spider Fusion

Spider fusion allows us to merge same colored spiders as long as they have one connection, adding their rotations together and connecting any inputs or outputs from the initial two spiders to the final spider.

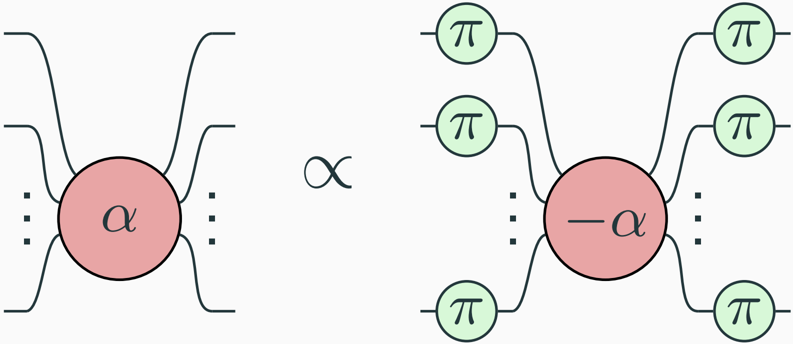


Hopf Rule

The hopf rule allows us to disconnect connections between opposite color spiders that come in pairs of two. A consequence of this with spider fusion is that *n* connections between opposite color spiders can always be considered to be equivalent to *n* mod 2 connections between those spiders.

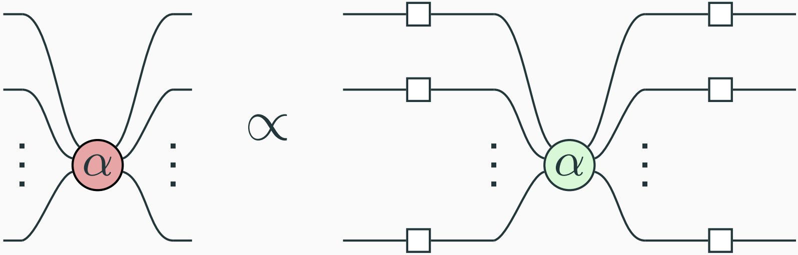
Bi-Pi Rule

The Bi-Pi rule allows us to add spiders of opposite colors with rotation π to every input and output of a spider and flip the phase.



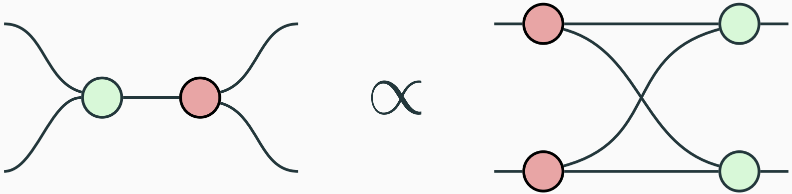
Bi-Hadamard Rule

The Bi-Hadamard rule allows us to add H-boxes to every input and output to flip the color of the spider within



Bialgebra rule

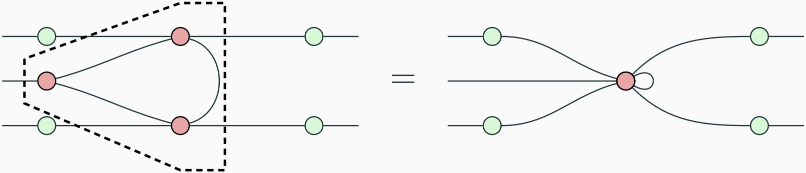
The bialgebra rule is unique in that it is one of the few rules that can introduce or remove swaps.



Identity Removal

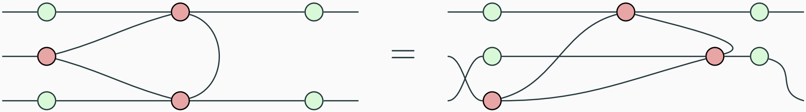
The identity removal rules allow us to remove spiders with $k2\pi$ rotations in general.

Fusion can be used between the three connected X spiders here to simplify our diagram.

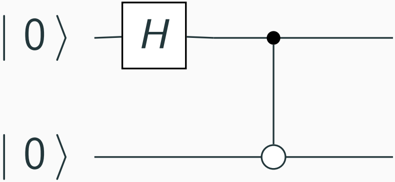


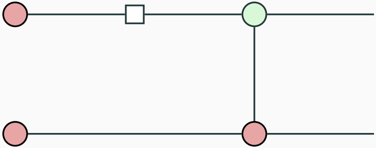
Only Connectivity Matters

We can freely move spiders around, as long as their connections and in/outputs remain the same



Example: Preparing a Bell Pair





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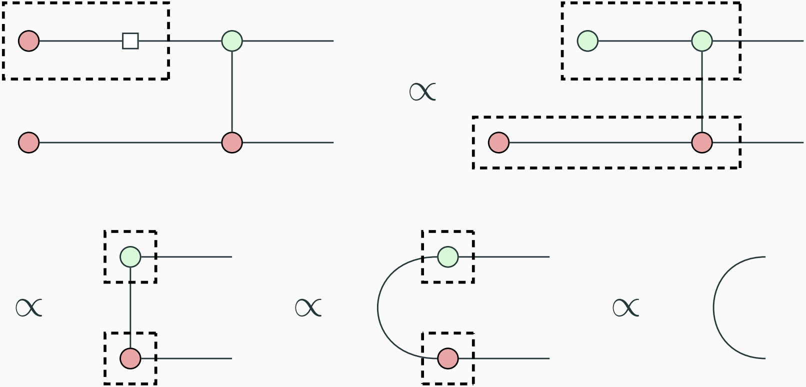
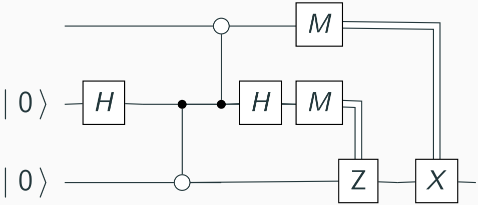
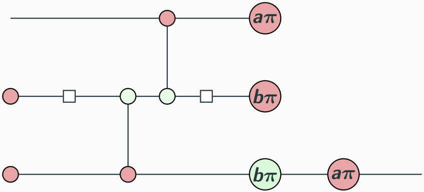


Example: Teleportation (Van de Wetering)





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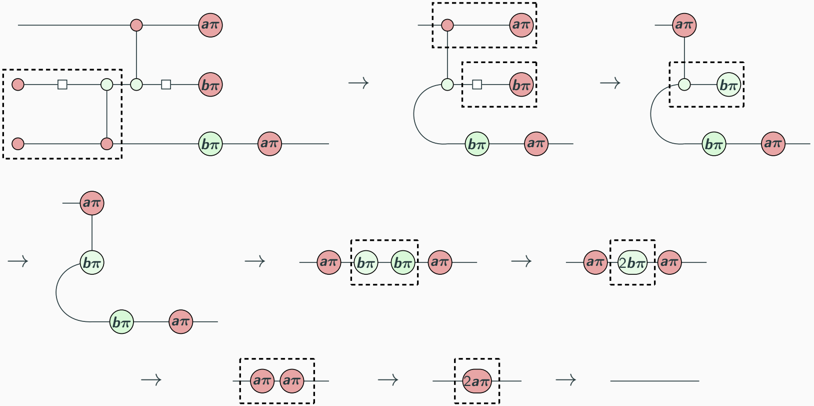


Reasoning Tools

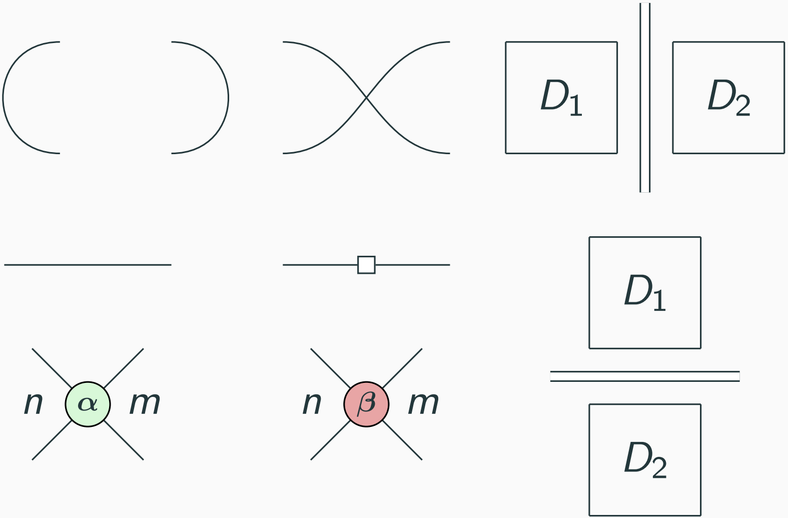
Unverified / Fully axiomatic:

- Quantomatic (https://quantomatic.github.io/)
- ZX Calculator (zx.cduck.me)
- Chyp

"Verify" by property testing:

- PyZX (https://github.com/Quantomatic/pyzx)

ZX Diagrams as string diagrams



Inductive ZX Diagrams

To define our ZX diagrams, we take these string diagram constructions and add Z and X spiders.

$$\frac{\mathsf{in}\ \mathsf{out} : \mathbb{N} \qquad \alpha : \mathbb{R}}{\texttt{Z\_Spider}\ \mathsf{in}\ \mathsf{out}\ \alpha : \texttt{ZX}\ \mathsf{in}\ \mathsf{out}} \qquad \frac{\mathsf{in}\ \mathsf{out} : \mathbb{N} \qquad \alpha : \mathbb{R}}{\texttt{X\_Spider}\ \mathsf{in}\ \mathsf{out}\ \alpha : \texttt{ZX}\ \mathsf{in}\ \mathsf{out}}$$

$$\frac{}{\texttt{Cap} : \texttt{ZX}\ 0\ 2} \qquad \frac{}{\texttt{Cup} : \texttt{ZX}\ 2\ 0} \qquad \frac{}{\texttt{Swap} : \texttt{ZX}\ 2\ 2} \qquad \frac{}{\texttt{Empty} : \texttt{ZX}\ 0\ 0}$$

$$\frac{\mathsf{zx1} : \texttt{ZX}\ \mathsf{in}\ \mathsf{mid} \qquad \mathsf{zx2} : \texttt{ZX}\ \mathsf{mid}\ \mathsf{out}}{\texttt{Compose}\ \mathsf{zx1}\ \mathsf{zx2} : \texttt{ZX}\ \mathsf{in}\ \mathsf{out}} \qquad \frac{}{\texttt{Wire} : \texttt{ZX}\ 1\ 1}$$

$$\frac{\mathsf{zx1} : \texttt{ZX}\ \mathsf{in1}\ \mathsf{out1} \qquad \mathsf{zx2} : \texttt{ZX}\ \mathsf{in2}\ \mathsf{out2}}{\texttt{Stack}\ \mathsf{zx1}\ \mathsf{zx2} : \texttt{ZX}\ (\mathsf{in1} + \mathsf{in2})\ (\mathsf{out1} + \mathsf{out2})} \qquad \frac{}{\texttt{Box} : \texttt{ZX}\ 1\ 1}$$

Semantics

To verify transformations on diagrams, we introduce a system of semantics. Our semantics system will rely on QuantumLib.

$$\texttt{Z\_Spider n m } \alpha \quad \mapsto \quad \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & e^{i\alpha} \end{bmatrix}$$

$$\texttt{X\_Spider n m } \alpha \quad \mapsto \quad H^{\otimes m} \times \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & e^{i\alpha} \end{bmatrix} \times H^{\otimes n}$$

More Semantics

$$
\begin{aligned}
\texttt{Cap} &\mapsto \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \\
\texttt{Cup} &\mapsto \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^\top \\
\texttt{Swap} &\mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\texttt{Empty} &\mapsto \begin{bmatrix} 1 \end{bmatrix} \\
\texttt{Wire} &\mapsto I_{2\times 2} \\
\texttt{Box} &\mapsto H \\
\texttt{Compose zx1 zx2} &\mapsto \texttt{semantics(zx2)} \times \texttt{semantics(zx1)} \\
\texttt{Stack zx1 zx2} &\mapsto \texttt{semantics(zx1)} \otimes \texttt{semantics(zx2)}
\end{aligned}
$$

Proportionality

Equivalence in ZX is up to constant factor

We define `proportionality` and use symbol $\propto$:

$$\exists c \neq 0 : \texttt{semantics}(zx1) = c * \texttt{semantics}(zx2) \implies zx1 \propto zx2$$

Allows Coq's rewriting capabilities in our proofs about diagrams

Why semantics?

- Smaller TCB
- Interoperability

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    Circuit structure is very different

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- Interoperability
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    - Circuit structure is very different
    - Convert circuit components
    - Prove equivalence through ground truth
- *Vy*ZX can calculate results

Three Proof Strategies

1. Proof through semantics

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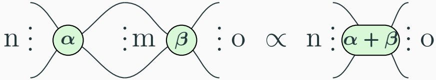
2. Inductive proof



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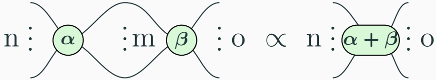
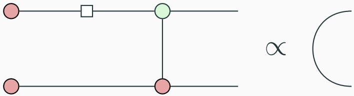


3. Diagrammatic proof



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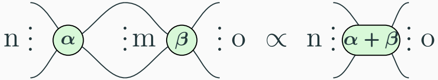
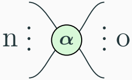
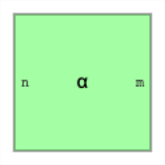


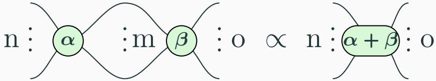
Other representation



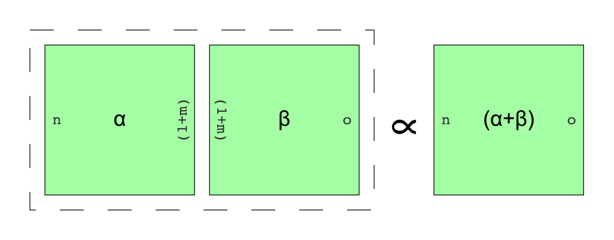
$$\widehat{=}$$



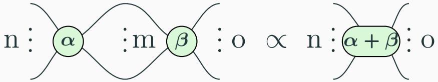
2. Inductive proof: Absolute Fusion



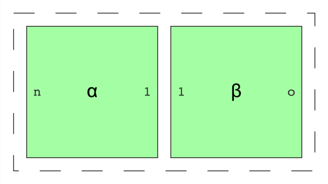
The lemma:



Let's induct over m

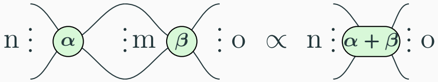


Base case:

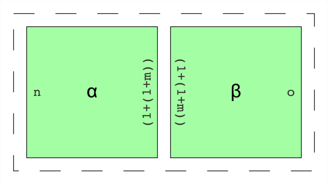


Solve with matrix semantics!

2. Inductive proof: Absolute Fusion



Inductive step:



Let's split out nodes to reduce size

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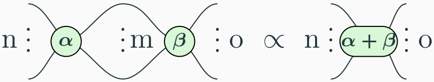
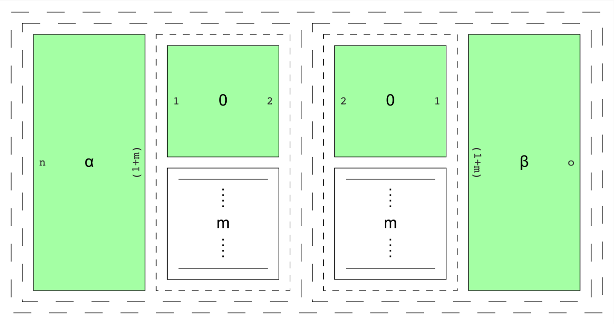
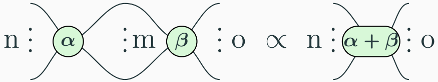


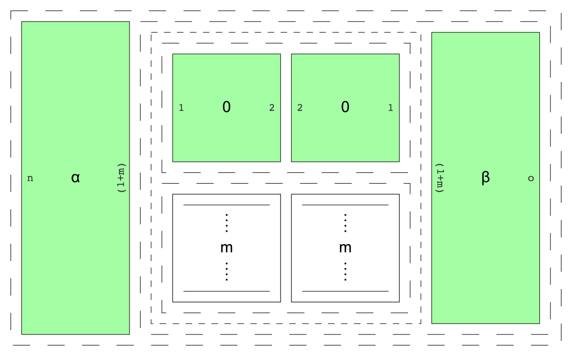


Idea: Fuse the small spiders. Problem: Association

2. Inductive proof: Absolute Fusion

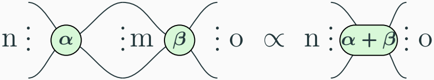


Reassociated diagram

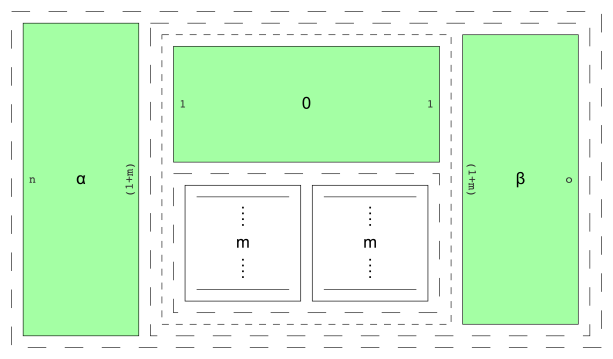


Let's fuse the small spiders

2. Inductive proof: Absolute Fusion



Upper spiders are fused



Apply identity rule, remove wires

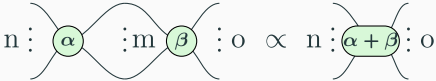
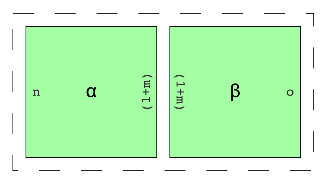


Diagram now corresponds to the IH



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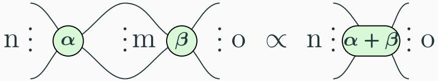
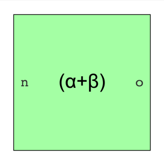
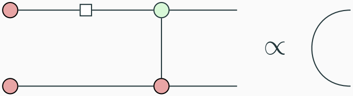


Qed.



3. Diagrammatic proof: Bell pair preparation



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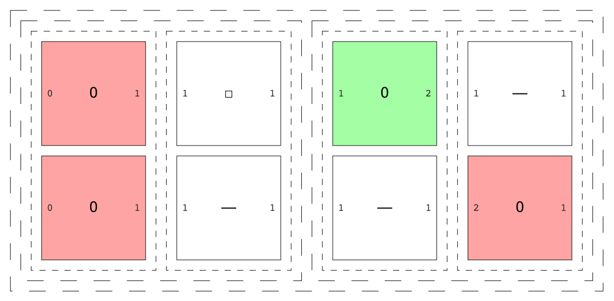
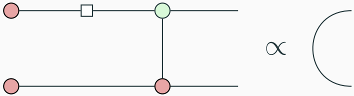
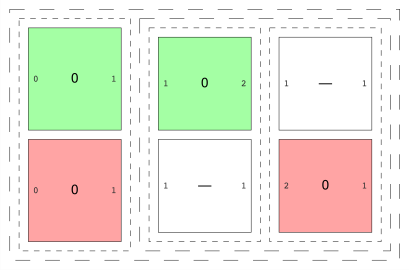


Let's first swap colors on the left

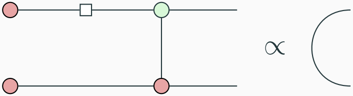
3. Diagrammatic proof: Bell pair preparation

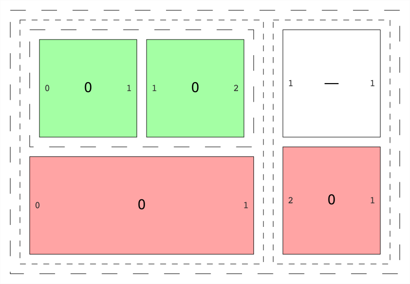




Let's reassociate!

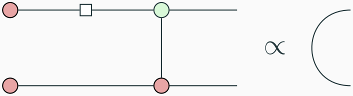
3. Diagrammatic proof: Bell pair preparation

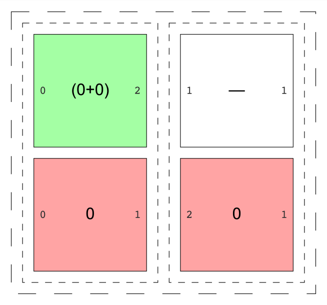




Let's fuse

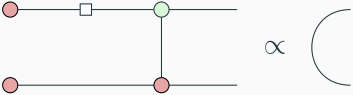
3. Diagrammatic proof: Bell pair preparation

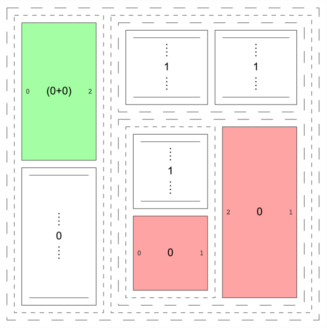




Separate the red node to fuse

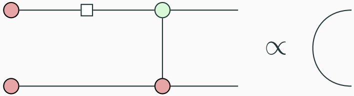
3. Diagrammatic proof: Bell pair preparation

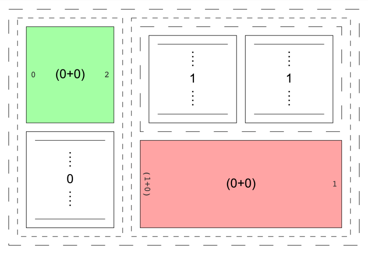




Let's fuse the red spiders

3. Diagrammatic proof: Bell pair preparation





Apply identity rule, remove wires

3. Diagrammatic proof: Bell pair preparation

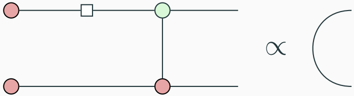
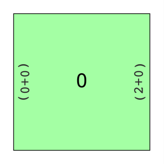
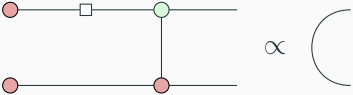


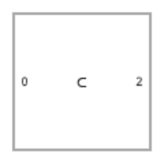
Diagram now corresponds to the a cap



3. Diagrammatic proof: Bell pair preparation



Qed.



Quantum circuit ingestion from RzQ gate set

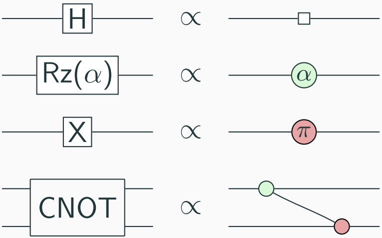
$$\frac{\mathsf{n} < \mathsf{q} \,:\, \mathbb{N}}{\texttt{H}\ \mathsf{n} \,:\, \mathsf{Circuit\ q\ q}} \qquad \frac{\mathsf{n} < \mathsf{q} \,:\, \mathbb{N}}{\texttt{X}\ n \,:\, \mathsf{Circuit\ q\ q}} \qquad \frac{\mathsf{n} < \mathsf{q} \,:\, \mathbb{N} \qquad \alpha \,:\, \mathbb{R}}{\texttt{Rz}(\alpha)\ n \,:\, \mathsf{Circuit\ q\ q}}$$

$$\frac{\mathsf{c,n} < \mathsf{q} \,:\, \mathbb{N}}{\texttt{CNOT}\ c\ n \,:\, \mathsf{Circuit\ q\ q}} \qquad \frac{\mathsf{q} \,:\, \mathbb{N} \qquad c_1, c_2 \,:\, \mathsf{Circuit\ q\ q}}{\texttt{Compose}\ c_1\ c_2 \,:\, \mathsf{Circuit\ q\ q}}$$

Quantum circuit ingestion from RzQ gate set

$$\frac{\mathsf{n} < \mathsf{q} : \mathbb{N}}{\mathtt{H}\ \mathsf{n} : \mathsf{Circuit\ q\ q}} \qquad \frac{\mathsf{n} < \mathsf{q} : \mathbb{N}}{\mathtt{X}\ n : \mathsf{Circuit\ q\ q}} \qquad \frac{\mathsf{n} < \mathsf{q} : \mathbb{N} \qquad \alpha : \mathbb{R}}{\mathtt{Rz}(\alpha)\ n : \mathsf{Circuit\ q\ q}}$$

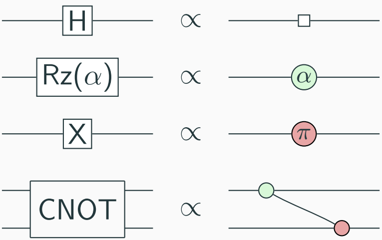
$$\frac{\mathsf{c,n} < \mathsf{q} : \mathbb{N}}{\mathtt{CNOT}\ c\ n : \mathsf{Circuit\ q\ q}} \qquad \frac{\mathsf{q} : \mathbb{N} \qquad c_1, c_2 : \mathsf{Circuit\ q\ q}}{\mathtt{Compose}\ c_1\ c_2 : \mathsf{Circuit\ q\ q}}$$

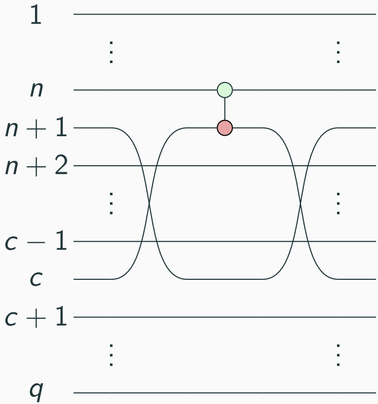


Quantum circuit ingestion from RzQ gate set

$$\frac{\mathsf{n} < \mathsf{q} \;:\; \mathbb{N}}{\mathtt{H}\ \mathsf{n} \;:\; \mathsf{Circuit\ q\ q}} \qquad \frac{\mathsf{n} < \mathsf{q} \;:\; \mathbb{N}}{\mathtt{X}\ n \;:\; \mathsf{Circuit\ q\ q}} \qquad \frac{\mathsf{n} < \mathsf{q} \;:\; \mathbb{N} \qquad \alpha \;:\; \mathbb{R}}{\mathtt{Rz}(\alpha)\ n \;:\; \mathsf{Circuit\ q\ q}}$$

$$\frac{\mathsf{c{,}n} < \mathsf{q} \;:\; \mathbb{N}}{\mathtt{CNOT}\ c\ n \;:\; \mathsf{Circuit\ q\ q}} \qquad \frac{\mathsf{q} \;:\; \mathbb{N} \qquad c_1, c_2 \;:\; \mathsf{Circuit\ q\ q}}{\mathtt{Compose}\ c_1\ c_2 \;:\; \mathsf{Circuit\ q\ q}}$$





Discussion

- Any ZX diagram can be expressed
- Multiple ways to encode
- Deal with associativity information
- Dimensionality issues

How can I verify my graphical language?

- Find underlying categorical structure
- Formally extend structure
- Translate into proof assistant
- Deal with resulting associtativity issues

Future work

- Restore connection information
- Verify ZX-based compiler
- Prove ZX results

Summary

- Defined ZX diagrams inductively
- Inspired by string diagrams
- Multiple proof strategies
- ZX calculus is interesting!

Find *Vy*ZX on GitHub

https://github.com/inQWIRE/VyZX

arXiv

Coming soon...

References

- Bob Coecke and Aleks Kissinger, *Picturing quantum processes: A first course in quantum theory and diagrammatic reasoning*, Cambridge University Press, 2017.
- Jonathan Castello, Patrick Redmond, and Lindsey Kuper, *Inductive diagrams for causal reasoning*, 2023.
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- John van de Wetering, *Zx-calculus for the working quantum computer scientist*, 2020.