*Vy*ZX

Formal Verification of a Graphical Language

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The ZX Calculus...

... is a graphical language for reasoning about quantum systems

ZX diagrams are open graphs consisting of green "Z" or red "X" "spiders" and "connections" between them

The Z and X spider

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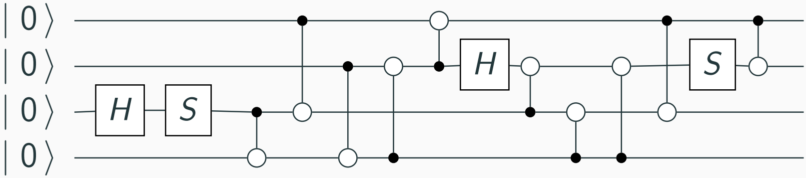
ZX diagrams are open graphs consisting of green “Z” or red “X” “spiders” and “connections” between them

The Z and X spider

Used for compilation, simulation, error correction & more

Key benefit: Diagrammatic rewrites complete & more comprehensible than circuits or matrices

Example: Entanglement (Van de Wetering)



Example: Entanglement (Van de Wetering)

$$[1\ \ 0\ \ 0\ \ 0\ \ i\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ i]^\top$$

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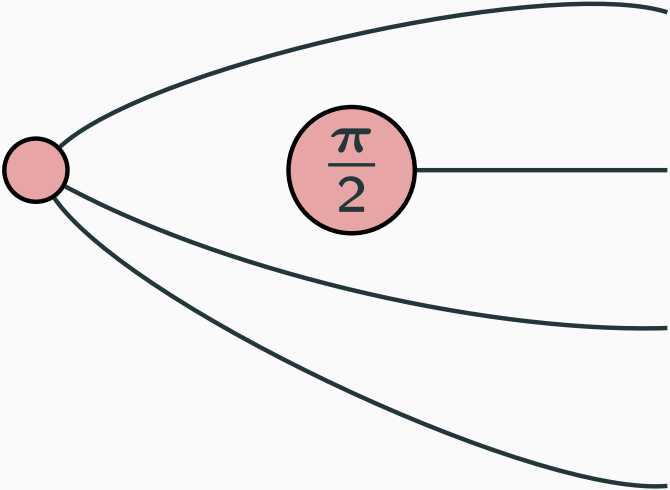
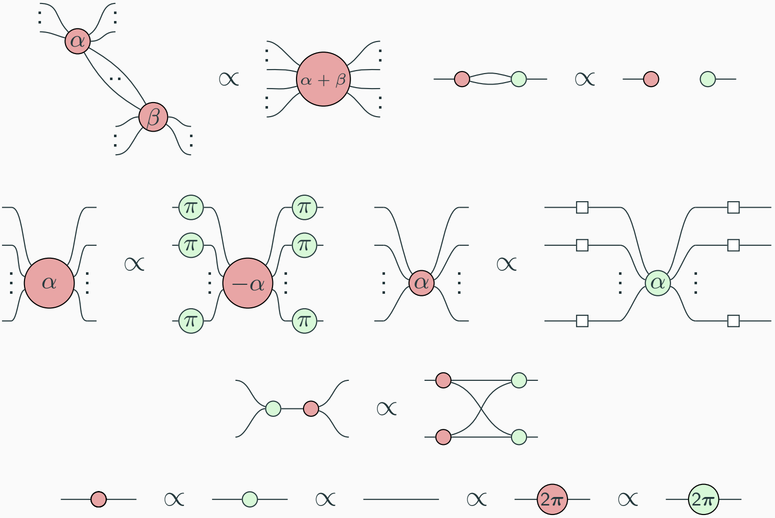


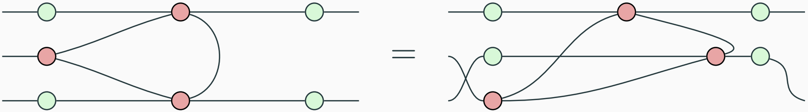
Rewriting Diagrams



Spider Fusion, The Hopf Rule, the Bi-pi Rule (Pi-Copy), Bi-hadamard Rule, the Bialgebra rule, and the identity rule

Only Connectivity Matters

We can freely move spiders around, as long as their connections and in/outputs remain the same



Reasoning Tools

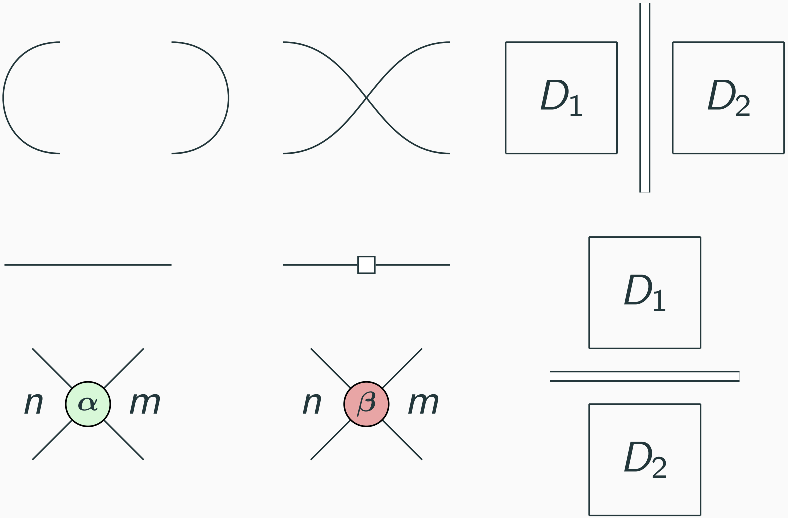
Unverified / Fully axiomatic:

- Quantomatic (https://quantomatic.github.io/)
- ZX Calculator (zx.cduck.me)
- Chyp

"Verify" by property testing:

- PyZX (https://github.com/Quantomatic/pyzx)

ZX Diagrams as string diagrams



Inductive ZX Diagrams

To define our ZX diagrams, we take these string diagram constructions and add Z and X spiders.

$$\frac{\text{in out} : \mathbb{N} \qquad \alpha : \mathbb{R}}{\texttt{Z\_Spider}\ \text{in out}\ \alpha : \texttt{ZX}\ \text{in out}} \qquad \frac{\text{in out} : \mathbb{N} \qquad \alpha : \mathbb{R}}{\texttt{X\_Spider}\ \text{in out}\ \alpha : \texttt{ZX}\ \text{in out}}$$

$$\frac{}{\texttt{Cap} : \texttt{ZX}\ 0\ 2} \qquad \frac{}{\texttt{Cup} : \texttt{ZX}\ 2\ 0} \qquad \frac{}{\texttt{Swap} : \texttt{ZX}\ 2\ 2} \qquad \frac{}{\texttt{Empty} : \texttt{ZX}\ 0\ 0}$$

$$\frac{\text{zx1} : \texttt{ZX}\ \text{in mid} \qquad \text{zx2} : \texttt{ZX}\ \text{mid out}}{\texttt{Compose}\ \text{zx1 zx2} : \texttt{ZX}\ \text{in out}} \qquad \frac{}{\texttt{Wire} : \texttt{ZX}\ 1\ 1}$$

$$\frac{\text{zx1} : \texttt{ZX}\ \text{in1 out1} \qquad \text{zx2} : \texttt{ZX}\ \text{in2 out2}}{\texttt{Stack}\ \text{zx1 zx2} : \texttt{ZX}\ (\text{in1} + \text{in2})\ (\text{out1} + \text{out2})} \qquad \frac{}{\texttt{Box} : \texttt{ZX}\ 1\ 1}$$

Semantics

To verify transformations on diagrams, we introduce a system of semantics. Our semantics system will rely on QuantumLib.

$$\texttt{Z\_Spider n m } \alpha \quad \mapsto \quad \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & e^{i\alpha} \end{bmatrix}$$

$$\texttt{X\_Spider n m } \alpha \quad \mapsto \quad H^{\otimes m} \times \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & e^{i\alpha} \end{bmatrix} \times H^{\otimes n}$$

More Semantics

$$
\begin{aligned}
\texttt{Cap} &\mapsto \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \\
\texttt{Cup} &\mapsto \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^\top \\
\texttt{Swap} &\mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\texttt{Empty} &\mapsto \begin{bmatrix} 1 \end{bmatrix} \\
\texttt{Wire} &\mapsto I_{2\times 2} \\
\texttt{Box} &\mapsto H \\
\texttt{Compose zx1 zx2} &\mapsto \texttt{semantics(zx2)} \times \texttt{semantics(zx1)} \\
\texttt{Stack zx1 zx2} &\mapsto \texttt{semantics(zx1)} \otimes \texttt{semantics(zx2)}
\end{aligned}
$$

Proportionality

Equivalence in ZX is up to constant factor

We define `proportionality` and use symbol $\propto$:

$$\exists c \neq 0 : \texttt{semantics}(zx1) = c * \texttt{semantics}(zx2) \implies zx1 \propto zx2$$

Allows Coq's rewriting capabilities in our proofs about diagrams

Why semantics?

- Smaller TCB
- Interoperability

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  We can ingest quantum circuits using sqir

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  Convert circuit components

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Why semantics?

- Smaller TCB
- Interoperability
  - We can ingest quantum circuits using sqir
  - Circuit structure is very different
  - Convert circuit components
  - Prove equivalence through ground truth
- *Vy*ZX can calculate results

Three Proof Strategies

1. Proof through semantics

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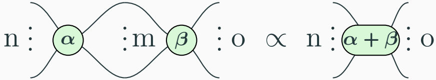
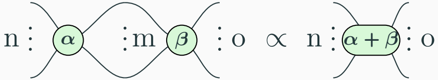
2. Inductive proof



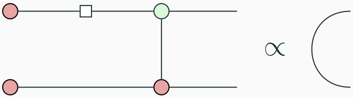
Three Proof Strategies

1. Proof through semantics

2. Inductive proof

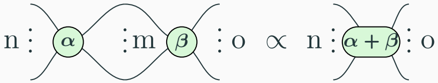


3. Diagrammatic proof

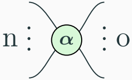


Three Proof Strategies

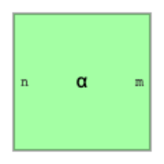
2. Inductive proof

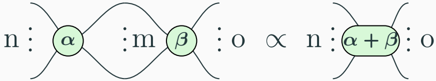


Other representation

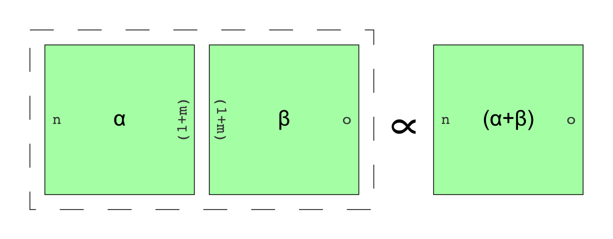


$$\widehat{=}$$



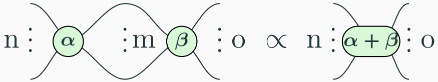


The lemma:

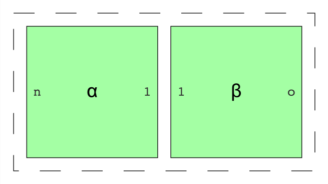


Let's induct over m

2. Inductive proof: Absolute Fusion

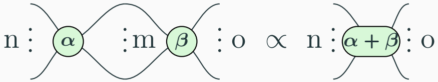


Base case:

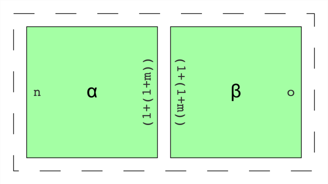


Solve with matrix semantics!

2. Inductive proof: Absolute Fusion

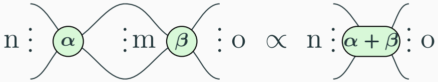


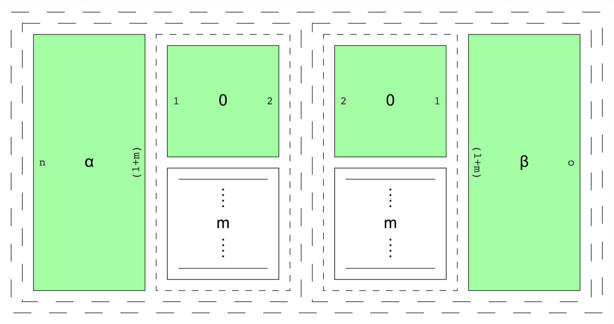
Inductive step:



Let's split out nodes to reduce size

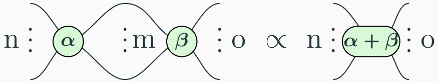
2. Inductive proof: Absolute Fusion



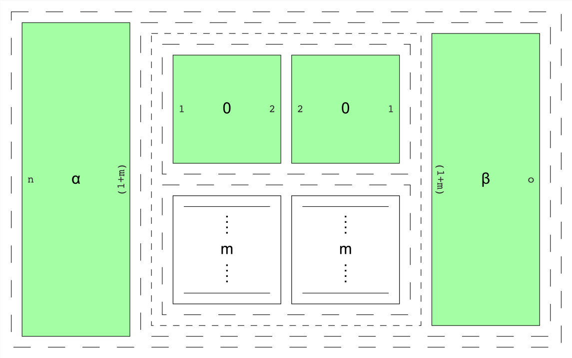


Idea: Fuse the small spiders. Problem: Association

2. Inductive proof: Absolute Fusion

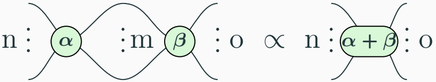


Reassociated diagram

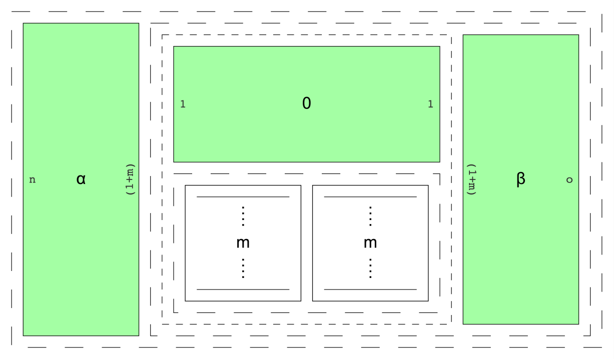


Let's fuse the small spiders

2. Inductive proof: Absolute Fusion



Upper spiders are fused



Apply identity rule, remove wires

2. Inductive proof: Absolute Fusion

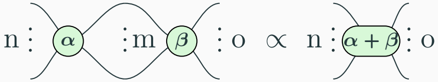
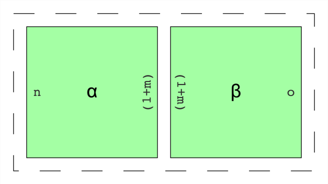
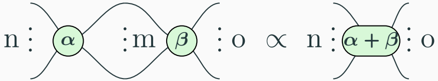


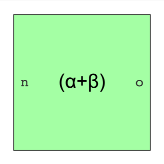
Diagram now corresponds to the IH



2. Inductive proof: Absolute Fusion



Qed.



Discussion

- Any ZX diagram can be expressed
- Multiple ways to encode
- Deal with associativity information
- Dimensionality issues

How can I verify my graphical language?

- Find underlying categorical structure
- Formally extend structure
- Translate into proof assistant
- Deal with resulting associtativity issues

Future work

- Restore connection information
- Verify ZX-based compiler
- Prove ZX results

Summary

- Defined ZX diagrams inductively
- Inspired by string diagrams
- Multiple proof strategies
- ZX calculus is interesting!

Find *Vy*ZX on GitHub

https://github.com/inQWIRE/VyZX

arXiv

Coming soon...

References

- Bob Coecke and Aleks Kissinger, *Picturing quantum processes: A first course in quantum theory and diagrammatic reasoning*, Cambridge University Press, 2017.
- Jonathan Castello, Patrick Redmond, and Lindsey Kuper, *Inductive diagrams for causal reasoning*, 2023.
- Kesha Hietala, Robert Rand, Shih-Han Hung, Xiaodi Wu, and Michael Hicks, *A verified optimizer for quantum circuits*, Proc. ACM Program. Lang. **5** (2021), no. POPL.
- John van de Wetering, *Zx-calculus for the working quantum computer scientist*, 2020.